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ELASTICITY OF DEMAND AND FLEXIBILITY OF PRICES

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The effective quantitative treatment of the theory of demand began with Cournot. His work was both critical and constructive. As critic he showed nearly a century ago the confusion and sterility of the current literary discussion of the theory of demand, which has persisted in many places even to the present time. As creative economist he introduced the conception of demand as a mathematical function of price;[1] he gave a mathematical definition of what has subsequently been described as elasticity of demand; and he showed how many problems have different solutions according as the demand for the article in question is elastic or inelastic. His work is a foundation upon which statistical economists can build.

COURNOT'S TREATMENT OF THE THEORY OF DEMAND

If $D=F(p)$ is the symbolic expression of the relation between the amount of commodity demanded, D, and the price per unit of commodity, p, then for many problems in economics it is of importance to know for what value of p the product $pF(p)$ is a maximum. One of Cournot's concrete illustrations is that of a monopolist, owning a mineral spring where the cost of production is negligible, who wishes to know what price of the commodity will yield him the largest monopoly return. The mathematical condition is

$$\frac{d[pF(p)]}{dp}=F(p)+pF'(p)=0 \quad . \quad . \quad . \quad \text{(i)}.$$

But in order to carry this problem to a concrete solution one must know the empirical form of $D=F(p)$. Upon this statistical problem Cournot makes the following comment:

> We may admit that it is impossible to determine the function $F(p)$ empirically for each article, but it is by no means the case that the same obstacles prevent the approximate determination of the value of p which satisfies equation (i) or which renders the product $pF(p)$ a maximum. *The construction of a table, where these values could be found, would be the work best calculated for preparing for the practical and rigorous solution of questions relating to the theory of wealth.*
>
> But even if it were impossible to obtain from statistics the value of p which should render the product $pF(p)$ a maximum, it would be easy to learn, at least for all articles

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[1] He even had the idea of the demand for any one commodity being a function of the prices of all commodities. *Researches into the Mathematical Principles of the Theory of Wealth,* chap. xi, §74.

to which the attempt has been made to extend commercial statistics, whether current prices are above or below this value. Suppose that when the price becomes $p+\Delta p$, the annual consumption as shown by statistics . . . becomes $D-\Delta D$. According as

$$\frac{\Delta D}{\Delta p} < \text{ or } > \frac{D}{p},$$

the increase in price, Δp, will increase or diminish the product $pF(p)$; and, consequently, it will be known whether the two values p and $p+\Delta p$ (assuming Δp to be a small fraction of p) fall above or below the value which makes the product under consideration a maximum.[1]

Commercial statistics should therefore be required to separate articles of high economic importance into two categories, according as their current prices are above or below the value which makes a maximum of $pF(p)$. We shall see that *many economic problems have different solutions according as the article in question belongs to one or the other of these two categories.*[2]

Some of Cournot's statements have been italicized in the preceding quotation to indicate that in his opinion we must know the empirical laws of demand for commodities if we are to pass to the practical and rigorous solution of questions relating to the theory of wealth.[3]

The classification that Cournot makes between commodities according as $\frac{\Delta D}{\Delta p} <$ or $> \frac{D}{p}$ is essentially the classification which has subsequently been made between inelastic and elastic commodities, flexible and inflexible prices. To show this, let us pass from Cournot's symbols to those that have been rendered more familiar by Professor Marshall. In Professor Marshall's notation, x stands for the amount demanded, and y, for the price per unit of commodity. If we substitute these symbols for those of Cournot we may write Cournot's statement in this way: According as $\frac{\Delta x}{\Delta y} <$ or $> \frac{x}{y}$, the increase in price, Δy, will increase or diminish the product xy.

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[1] The method of reaching the inequality discussed in the text may be indicated:
The increase in price will increase the gross receipts if

(1) $(p+\Delta p)(D-\Delta D) > pD$, or
(2) $pD - p\cdot\Delta D + D\cdot\Delta p - \Delta p\cdot\Delta D > pD$, or
(3) $-p\cdot\Delta D + D\cdot\Delta p - \Delta p\cdot\Delta D > 0$, or
(4) $\Delta D(p+\Delta p) < D\cdot\Delta p$, or
(5) $\frac{\Delta D}{\Delta p} < \frac{D}{p+\Delta p}$, or, when Δp is small as compared with p,
(6) $\frac{\Delta D}{\Delta p} < \frac{D}{p}$.

[2] Cournot, *Researches into the Mathematical Principles of the Theory of Wealth*, pp. 53–54.

[3] Stanley Jevons seems to have had similar thoughts when he was engaged upon his unfinished *Principles of Economics*. He says on pp. 146–7: "It would be a matter of great importance, if it were practicable, to ascertain statistically the exact law of variation of price of the more important commodities. Assuming the demand to be constant, in the sense that there is a constant population of purchasers with fixed tastes, we should make the supply of the commodity—say wheat, sugar, tea—the variable, and then ascertain the changes of price, the variant. . . . As to the value of the laws, if we could determine them, there is little doubt. They would be practically important to merchants in judging of the probable effect of changes in supply; they would be theoretically important in giving reality to the theory of exchange, and allowing us actually to trace out the effects of all kinds of variations in trade.

"The difficulties in the way of such empirical determinations of laws are so formidable that I entertain little hope of successful investigations being made for many years to come."

So far Cournot carried the problem. But his statement may be easily simplified and its economic significance rendered much clearer. In Figure 1 we have the familiar graph of the law of demand, where $x =$ the amount of the commodity demanded; $y =$ the price per unit of commodity; and DD' is the demand curve. Let the amount actually demanded be $x = OM$ and the resulting price be $y = MP$. Then, following Cournot's formula, it would be profitable to the monopolist to raise or lower his price according as

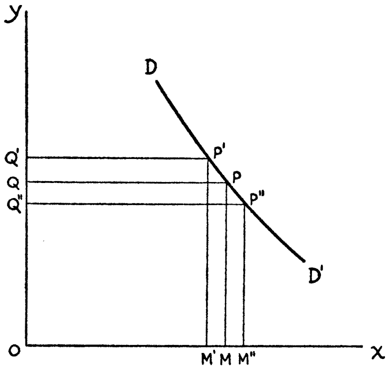

FIGURE 1

$$\frac{\Delta x}{\Delta y} < \text{ or } > \frac{x}{y} \quad . \; . \; . \quad \text{(ii)}.$$

The inequality (ii) may be divided through either by $\frac{x}{y}$ or by $\frac{\Delta x}{\Delta y}$. If we divide through by $\frac{x}{y}$, we get

$$\frac{\Delta x}{\Delta y} \cdot \frac{y}{x} < \text{ or } > 1 \quad . \; . \; . \quad \text{(iii)},$$

which is Professor Marshall's form of statement. He regards $\frac{\Delta x}{\Delta y} \cdot \frac{y}{x}$ as the measure of the elasticity of demand, and describes the demand for the commodity as inelastic or elastic according as $\frac{\Delta x}{\Delta y} \cdot \frac{y}{x}$ is less or greater than unity. If we call $\frac{dx}{dy} \cdot \frac{y}{x}$ the coefficient of elasticity of demand and indicate it by η, we may state Cournot's proposition as follows: The gross receipts of a monopolist will be increased or diminished by an increase in price according as $\eta <$ or > 1, that is, according as the demand for the commodity is inelastic or elastic. This form of expression we have reached by dividing Cournot's inequality (ii) by $\frac{x}{y}$.

But we might equally well have divided it by $\frac{\Delta x}{\Delta y}$. If we do so, we shall have

$$1 < \text{ or } > \frac{x}{y} \cdot \frac{\Delta y}{\Delta x} \quad . \quad . \quad . \quad \text{(iv).}$$

Let us now call $\frac{dy}{dx} \cdot \frac{x}{y}$ the coefficient of the flexibility of prices and indicate it by ϕ. Cournot's proposition would then be as follows: The gross receipts of a monopolist will be increased or diminished by an increase in price according as $\phi >$ or < 1, that is, according as prices are flexible or inflexible.

The relation between ϕ and η is simply

$$\phi = \frac{1}{\eta} \quad . \quad . \quad . \quad \text{(v).}$$

If we can find statistically either ϕ or η, equation (v) will enable us to find the other.

TYPICAL EQUATION TO THE LAW OF DEMAND

In a later section of this paper a method will be presented by means of which the empirical laws of demand may be ascertained, but before passing to the statistical precedure it is desirable to know what the typical forms of the demand curve must be. Here we follow a hint that is given by Cournot:

> If we cease considering the question from an exclusively abstract standpoint, it will be instantly recognized how improbable it is that the function $pF(p)$ should pass through several intermediate maxima or minima inside of the limits between which the value of p can vary; and as it is unnecessary to consider maxima which fall beyond these limits, if any such exist, all problems are the same as if the function $pF(p)$ only admitted a single maximum. The essential question is always whether, for the extent of the limits of oscillation of p, the function $pF(p)$ is increasing or decreasing for increasing values of p.[1]

The purport of Cournot's reasoning we may examine a little more closely. Let the law of demand be $y = F(x)$ where y is the price per unit of commodity and x is the amount of the commodity demanded at price y. The maximum gross receipts will be obtained when $xF(x)$ is a maximum. According to Cournot, practical experience teaches that as a rule y varies only within comparatively narrow limits and that, consequently, for practical purposes we may assume that at only one value of x is $xF(x)$ a maximum. We know, however, that when $xF(x)$ reaches a maximum both the elasticity of demand and the flexibility of prices are equal to unity[1] and that before the maximum is reached demand is elastic and prices are inflexible, while just after the point of the maximum the demand is inelastic and prices are flexible. This fact gives the clue to the method of deriving the typical equation to the demand curve.

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[1] Cournot, *op. cit.*, p. 55.

The simplest method of expressing symbolically the Cournot hypothesis is to regard either the elasticity of demand or the flexibility of prices as a linear function of the amount demanded. In case of the flexibility of prices we may write the assumption as follows:

$$\phi = \alpha + \beta x \quad . \quad . \quad . \quad \text{(vi)}.$$

But according to the definition of flexibility of prices, $\phi = \frac{dy}{dx} \cdot \frac{x}{y}$ which may be written $\phi = \frac{dy}{y} \Big/ \frac{dx}{x}$. We may, therefore, put equation (vi) in this form:

$$\frac{dy}{y} \Big/ \frac{dx}{x} = \alpha + \beta x, \text{ or}$$

$$\frac{dy}{y} = \alpha \frac{dx}{x} + \beta dx.$$

Integrating, we get

$$\log_e y = \alpha \log_e x + \beta x + \log_e A.$$

Or, passing from logarithms to absolute numbers,

$$y = A x^{\alpha} e^{\beta x} \quad . \quad . \quad . \quad \text{(vii)}.$$

Equation (vii) may be called the typical equation to the law of demand.

The typical equation to the law of demand has been reached by giving an accurate symbolic definition of flexibility of prices and then following out the implication of Cournot's discussion as to the nature of the law of demand with respect to the variation of the flexibility of prices. A comparison of equation (vi) and (vii) shows that if the typical equation $y = A x^{\alpha} e^{\beta x}$ is ascertained empirically from statistical data we may write at once the equation to the flexibility of prices $\phi=\alpha+\beta x$, because the two constants α and β in $\phi=\alpha+\beta x$ are explicitly given in $y=Ax^{\alpha}e^{\beta x}$.

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[1] The proof is quite simple. If $y = F(x)$, then $xF(x)$ is a maximum when $\frac{d[xF(x)]}{dx} = 0$, that is, when $F(x) + xF'(x) = 0$, or when $y + x\frac{dy}{dx} = 0$. This last condition may be put in the form $\frac{x}{y} \cdot \frac{dy}{dx} = -1$. But $\frac{x}{y} \cdot \frac{dy}{dx} = \phi$ and by equation (v), $\phi = \frac{1}{\eta}$.

Equation (vii) was reached on the assumption that the flexibility of prices is a simple linear function of x. We may make the hypothesis a little more complex by assuming that

$$\phi=\alpha+\beta x+\gamma x^2 \quad . \quad . \quad . \quad \text{(viii)}.$$

Then, since by definition $\phi=\frac{dy}{y}\Big/\frac{dx}{x}$, we have

$$\frac{dy}{y}=\alpha\frac{dx}{x}+\beta dx+\gamma x dx.$$

Integrating, we get

$$\log_e y=\alpha \log_e x+\beta x+\frac{\gamma x^2}{2}+\log_e A,$$

or, passing from logarithms to absolute quantities,

$$y=Ax^{\alpha}e^{\beta x+\frac{\gamma x^2}{2}} \quad . \quad . \quad . \quad \text{(ix)}.$$

This gives the typical equation to the law of demand in a slightly more complex form.

PROCEDURE IN THE STATISTICAL DERIVATION OF THE LAW OF DEMAND

The *Yearbook of the Department of Agriculture* gives statistics of the production and of the December farm prices of agricultural commodities throughout a long interval. The data referring to the production and prices of potatoes from 1881 to 1913 will be used to illustrate the derivation of the empirical law of demand for potatoes and of the law of the variation of the flexibility of prices or of the elasticity of demand.

In Figure 2 the secular trend of the production of potatoes from 1881

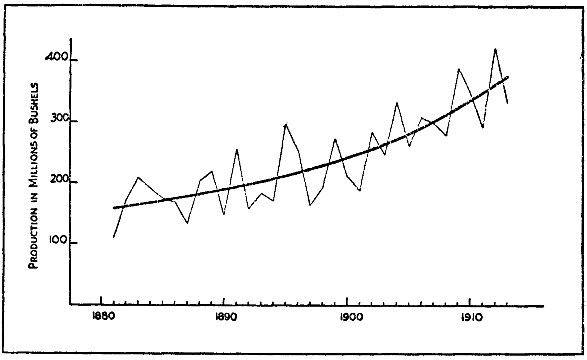

FIGURE 2. THE SECULAR TREND IN THE PRODUCTION OF POTATOES IN THE UNITED STATES. $y=222.3+5.711t+.1758t^2+.004363t^3$, ORIGIN AT 1897

to 1913 is graphed, and a similar curve for the secular trend of the prices of potatoes might be given. The equation to the trend of production is printed on Figure 2; the equation to the trend of prices is, with origin at 1897,

$$y = 48.86 + .775\,t + .0443\,t^2 - .002935\,t^3.$$

After the trends of the two series of figures—production and prices—have been computed, the next step in the problem is to express the observed values of the quantities as ratios to the corresponding respective trends. The ratios relating to production and prices we shall refer to, respectively, as production ratios and price ratios. In Table I these ratios are tabulated.

The correlation of the price ratios and the production ratios of potatoes is $r = -.84$, which is sufficient evidence of a very high relation between the two series. Our problem now is to find for any one year the relation between the variation of the amount of the commodity

TABLE I

THE ANNUAL PRODUCTION OF POTATOES AND THEIR DECEMBER FARM PRICES IN THE UNITED STATES. PRODUCTION RATIOS AND PRICE RATIOS

Year	Production: millions of bushels x	Price: cents per bushel y	Production ratio X	Price ratio Y
1881	109	91.0	.690	1.52
82	171	55.7	1.062	.97
83	208	42.2	1.261	.77
84	191	39.6	1.137	.74
85	175	44.7	1.023	.88
86	168	46.7	.960	.94
87	134	68.2	.753	1.41
88	202	40.2	1.110	.85
89	218	35.4	1.172	.75
1890	148	75.8	.783	1.63
91	254	35.8	1.316	.76
92	157	66.1	.793	1.41
93	183	59.4	.906	1.27
94	171	53.6	.826	1.13
95	297	26.6	1.401	.56
96	252	28.6	1.161	.59
97	164	54.7	.739	1.12
98	192	41.4	.842	.84
99	273	39.0	1.162	.77
1900	211	43.1	.876	.84
01	188	76.7	.758	1.46
02	285	47.1	1.113	.88
03	247	61.4	.936	1.13
04	333	45.3	1.224	.82
05	261	61.7	.929	1.09
06	308	51.1	1.058	.89
07	298	61.8	.990	1.06
08	279	70.6	.894	1.21
09	389	54.1	1.201	.90
1910	349	55.7	1.039	.93
11	293	79.9	.840	1.33
12	421	50.5	1.163	.83
13	332	68.7	.883	1.14
Total			33.001	33.42
Mean			1.00	1.01

produced and the corresponding most probable price, and we have to solve the problem from our knowledge of the relation between production ratios and price ratios throughout an interval of time.

Let X represent the production ratio and Y the price ratio, and let us assume that the typical equation connecting X and Y is

$$Y = A'X^{\alpha'}e^{\beta' X} \quad . \; . \; . \quad \text{(x)}.$$

The reason for assuming this type of equation will appear at the end of the article. Before proceeding to fit equation (x) to the data, we may effect a simplification by observing that the value of A' is already known. When X, the price ratio, is 1.0, the production ratio is likewise 1.0, and if these values[1] are substituted in (x), the value of A' is found to be $e^{-\beta'}$,

and consequently (x) becomes

$$Y = X^{\alpha'}e^{\beta'(X-1)} \quad . \; . \; . \quad \text{(xi)}.$$

To fit equation (xi) to the observations, let us first take logarithms of both sides of the equation. We have

$$\log Y = \alpha' \log X + \beta'(X-1) \log e \quad . \; . \; . \quad \text{(xii)}.$$

If the method of least squares is used to fit equation (xii) to the observations, we have as the observation equations

$$[\log Y_1 - \alpha' \log X_1 - \beta'(X_1 - 1) \log e]^2 = v^2_1$$

$$. \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; .$$

$$[\log Y_N - \alpha' \log X_N - \beta'(X_N - 1) \log e]^2 = v^2_N.$$

The sum of the squares of the errors, $\Sigma(v^2)$ is a function of α' and β', and in order to find their most probable values, we have the normal equations

$$\frac{\partial \Sigma(v^2)}{\partial \alpha'} = \Sigma(\log Y \log X) - \alpha' \Sigma(\log X)^2 - .4343\,\beta' \Sigma(X \log X) + .4343\,\beta' \Sigma(\log X) = 0;$$

$$\frac{\partial \Sigma(v^2)}{\partial \beta'} = \Sigma(X \log Y) - \Sigma(\log Y) - \alpha' \Sigma(X \log X) + \alpha' \Sigma(\log X) - .4343\,\beta' \Sigma(X^2) + .8686\,\beta' \Sigma(X) - .4343\,N\beta' = 0.$$

By means of these two normal equations the most probable values of α' and β' may be determined from the observations. When the calculation is carried out for the production ratios and price ratios of potatoes, the equation connecting the two is found to be

$$Y = X^{.143}e^{-1.376(X-1)} \quad . \; . \; . \quad \text{(xiii)}.$$

Its graph is drawn in Figure 3.

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[1] In the case of potatoes the mean values of X, Y are, practically, 1.0. See Table I. There may be cases where it would be better not to use this hypothesis but to determine A' from the observations. This could be done by following the same method as that described in the text.

Equation (xiii) gives the relation between the price ratios and the production ratios for the interval 1881–1913. It may be called the law of demand for potatoes in the ratio form. A simple transformation will make possible the next step, which is the derivation of the law of demand for any one year in terms of absolute quantities. Let T_x be the trend of production for any given year and T_y be the corresponding trend of prices for the same year. According to the definition of X and Y as they appear in equation (xiii), $X = \frac{x}{T_x}$; $Y = \frac{y}{T_y}$. By making these substitutions in (xiii), the law of demand for potatoes in the absolute form becomes

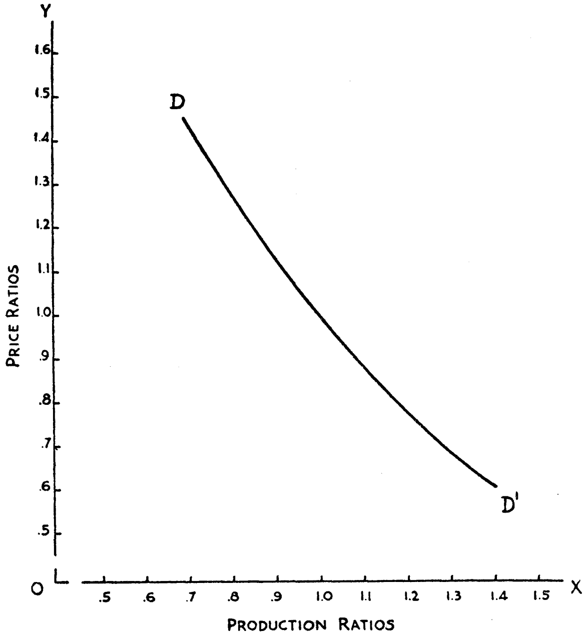

FIGURE 3. THE LAW OF DEMAND FOR POTATOES. (RATIO FORM.)
$Y = X^{.143} e^{-1.376\,(X-1)}$

$$y = \frac{T_y}{T_x^{.143}} e^{1.376} x^{.143} e^{-\frac{1.376}{Tx}x} \quad . \ . \ . \quad \text{(xiv)},$$

$$= A x^{.143} e^{-\frac{1.376}{Tx}x}, \text{ where } A = \frac{T_y}{T_x^{.143}} e^{1.376}.$$

If, for example, the law of demand for potatoes in 1913 is to be ascertained, find the trend of prices for 1913, T_y, and the trend of production for 1913, T_x, and substitute these values in (xiv). The values of

T_y, T_x may be obtained for any one year from the equations to the trends which have already been given.[1]

After the law of demand, in the absolute form, has been ascertained it is a simple matter to find the law of the variation of the flexibility of prices, or of the elasticity of demand. Since ϕ, the flexibility of prices, is by definition equal to $\frac{dy}{y} \Big/ \frac{dx}{x} = \frac{x}{y} \cdot \frac{dy}{dx}$, all that is needed is to find this value from equation (xiv). But this process is not necessary, for it may be shown that the equation descriptive of ϕ may be written immediately from equation (xiii), that is to say, the equation descriptive of the flexibility of prices (or of the elasticity of demand) may be inferred immediately from the law of demand in the ratio form (equation xiii). In this particular case $\phi = .143 - 1.376\ X$, and the manner in which the constants are obtained from (xiii) is obvious. Later on a general proof will be given of the legitimacy of deriving the equation to the flexibility of prices in this way. In the meantime we shall examine a little more in detail the information given by the equation $\phi = .143 - 1.376\ X$. Its graph is traced in Figure 4, and we are able to see at a

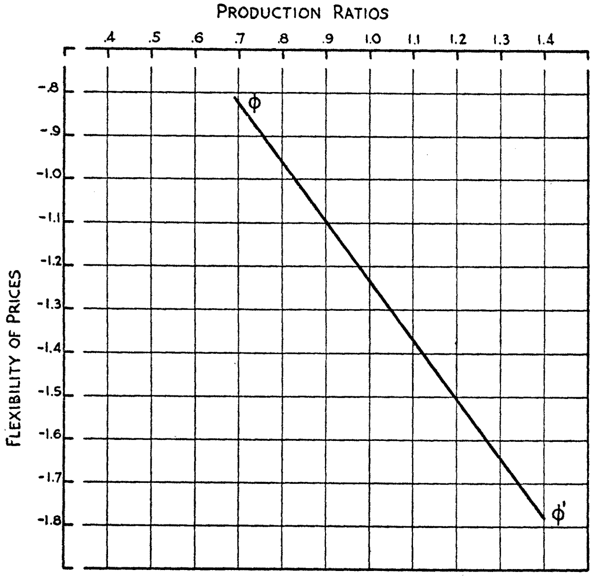

FIGURE 4. THE FLEXIBILITY OF THE PRICES OF POTATOES.
$\phi = .143 - 1.376\ X$

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[1] The objection may be made that since the trends are described by parabolas of the type, $y = a + bt + ct^2 + dt^3$, if we extrapolate for many years beyond the observations the curves may give impossible results. But the subject of the present article is not what will occur many years beyond the limits of observation. To a similar objection on the part of an unfriendly critic Pareto once replied: *Quando vi si dà una formula valevole entro certi limiti, chi vi insegna ad applicarla fuori di quei limiti?*

glance how the flexibility of prices varies from point to point in the demand curve. The maximum value of ϕ is $-.81$ and its minimum value is -1.78. Consequently, since η, the elasticity of demand, is related to ϕ through the equation $\eta = \frac{1}{\phi}$, we know that, in the case of potatoes, for this particular market,[1] the maximum value of η is $-\frac{1}{1.78}$, and its minimum value is $-\frac{1}{.81}$.

This method of calculating ϕ and η ought to throw considerable light upon a perplexing statement by Professor Marshall in his classic chapter on "The Elasticity of Wants":

> The elasticity of demand is great for high prices, and great, or at least considerable for medium prices; but it declines as the price falls; and gradually fades away if the fall goes so far that satiety level is reached. This rule appears to hold with regard to nearly all commodities and with regard to the demand of every class.[2]

I have called this statement perplexing because we have had no means of deriving empirical laws of demand and because it has been impossible to picture concretely what is meant by elasticity of demand being "great" or "considerable." One knows when a coefficient of correlation is "great" or "considerable," but what is a "great" or a "considerable" elasticity of demand? The graph of the values of ϕ or η gives a definite idea.

Another matter upon which the equation $\phi = \alpha' + \beta' X$, or, in this particular case, $\phi = .143 - 1.376\, X$, throws light is the problem of ascertaining for what value of x, the gross receipts, xy are a maximum. We know that when the gross receipts are a maximum, the flexibility of prices, or the elasticity of demand, is equal to -1. All that is necessary to find the proper value of x is to take the following two steps. First place $\phi = -1$ in the equation $\phi = \alpha' + \beta' X$, or, in this particular case, $\phi = .143 - 1.376\, X$ and solve for X. This gives the solution in terms of ratio of production to trend of production. In the particular case before us $X = .83$; that is to say, according to the experience from 1881 to 1913, the gross receipts of the farmer from the sale of potatoes reached a maximum when the production fell 17 per cent short of the trend of production. If it is now desired to pass from percentages to absolute quantities, we may take the next step and put $X = \frac{x}{T_x} = .83$ and solve for x, which will give the absolute number of bushels that will probably give the maximum return in the year for which T_x is the trend of production.

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[1] It is always necessary to specify the market for which the empirical laws of demand apply. The values of ϕ and η vary from market to market.

[2] Marshall, *Principles of Economics*, 4th edit., p. 178.

We return now to the method of deriving the equation descriptive of ϕ, the flexibility of prices. In the earlier part of this paper we assumed that the typical equation descriptive of the relation between price ratios and production ratios was $Y=A'X^{\alpha'}e^{\beta'}X$ (see equation (x)). We proved that the typical equation in the absolute form was $y=Ax^{\alpha}e^{\beta x}$ (see equation (vii)), on the assumption that $\phi=\alpha+\beta x$. We shall now show that $\alpha=\alpha'$ and $\beta=\frac{\beta'}{T_x}$, and consequently, when the statistical work has been carried as far as the derivation of the law of demand in the ratio form $Y=A'X^{\alpha'}e^{\beta' X}$, we can write immediately the equation to ϕ, namely,

$$\phi=\alpha'+\frac{\beta'}{T_x}x=\alpha'+\beta' X.$$

In the equation $Y=A'X^{\alpha'}e^{\beta' X}$ put $Y=\frac{y}{T_y}$, $X=\frac{x}{T_x}$. Then,

$$\frac{y}{T_y}=A'\left(\frac{x}{T_x}\right)^{\alpha'}e^{\beta'\left(\frac{x}{T_x}\right)}, \text{ consequently, } y=A'\frac{T_y}{T_x^{\alpha'}}x^{\alpha'}e^{\frac{\beta'}{T_x}x}$$

$$=Ax^{\alpha'}e^{\frac{\beta'}{T_x}x}, \text{ where } A=A'\frac{T_y}{T_x^{\alpha'}}.$$

Taking logarithms of both sides of $y=Ax^{\alpha'}e^{\frac{\beta'}{T_x}x}$,

$$\log y=\log A+\alpha'\log x+\frac{\beta'}{T_x}x,$$

and then differentiating we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{\alpha'}{x}+\frac{\beta'}{T_x}, \text{ whence, } \frac{x}{y}\frac{dy}{dx}=\alpha'+\frac{\beta'}{T_x}x.$$

But $\frac{x}{y}\frac{dy}{dx}=\phi$ according to definition, and we may write

$$\phi=\alpha'+\frac{\beta'}{T_x}x=\alpha'+\beta' X.$$

The equation description of the flexibility of prices may, therefore, be written immediately from the law of demand in the ratio form

$$Y=A'X^{\alpha'}e^{\beta' X}.$$